ON MASS AND HEAT TRANSFER FROM A SPHERICAL PARTICLE IN A LAMINAR STREAM OF VISCOUS FLUID

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An approximate solution is obtained for the problem of mass and heat transfer from a moving rigid spherical particle at small finite values of Péclet and Reynolds number. The case of a first-order chemical reaction of arbitrary speed at the surface of the particle is considered. The problem is solved by the method of matched asymptotic expansions with respect to Péclet number. The concentration and temperature fields are constructed, and the complete flux of material and heat at the surface of the particle is found.

The basic equation in the problem is the equation of convective diffusion or heat conduction, in which the velocity field of the viscous flow is assumed known from the solution of the corresponding hydrodynamic problem. The dispersion of fine aerosol particles due to Brownian motion is also governed by an analogous equation.

The diffusive flux on a spherical particle under conditions of Stokes flow, that is for the Reynolds number $R \to 0$, was calculated in the approximation of a diffusion boundary layer in [1]. In paper [2] the Stokes distribution of velocity was replaced by the solution obtained in [3] by the method of matched asymptotic expansions, and in the approximation of a diffusion boundary layer an analytic expression was found for the diffusive flux on a sphere at finite Reynolds number, which agrees with the results of [1] as $R \to 0$. Comparison of the results [2] with a numerical solution of the problem [4] shows that the analytical solution agrees well with the exact solution up to $R \approx 10$.

The approximation of a diffusion boundary layer limits the region of applicability of the results [1, 2, 4] to Péclet numbers $P \gg 1$. Attempts to solve the problem for small finite Péclet numbers have been undertaken repeatedly. However only in [5] has a solution been obtained by the method of matched asymptotic expansions with respect to Péclet number for Stokes flow. That paper also gives references to earlier works, the results of which proved to be erroneous. By virtue of the assumptions adopted in [5] the results are valid in the regime of pure diffusion for Reynolds number $R \to 0$.

In the present work the problem of diffusion to a spherical particle at finite Péclet number is extended to the case of finite Reynolds number and chemical reaction at the surface of the particle. Extension of the range of Reynolds number is achieved by using for the velocity field the expression in [3], giving the flow past a spherical particle at finite R. An attempt at solving the problem in the special case of infinitely high speed of chemical reaction is given in [6], but as a result of errors committed by the author in matching, the results he obtained are incorrect.

1. Statement of problem. Method of solution. We consider the steady process of diffusion in a stream of viscous incompressible fluid flowing past a rigid spherical particle of radius a. Far from the sphere the speed of the flow is U, and the concentration of the diffusing component is c_0.

In the vicinity of the sphere the speed of the stream and the concentration of diffusing material vary, the speed because of the perturbing action of the sphere and the concentration because of absorption of material at its surface. The velocity field is assumed known from the solution of the corresponding hydrodynamic problem. The object of the present work is the determination of the concentration field and of its most important feature, the flux of matter at the sphere.

We write the equation of convective diffusion appropriate to the problem under consideration, using dimensionless variables and the system of coordinates shown in Fig. 1

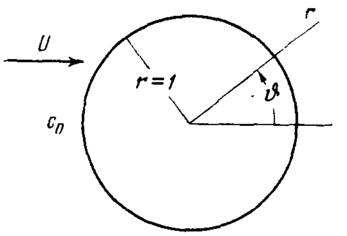

Fig. 1

$$\Delta\xi = \frac{P\partial(\psi, \xi)}{r^2\partial(r, \mu)}, \quad \xi = \frac{c_0 - c}{c_0},$$

$$P = \frac{Ua}{D} \quad (\mu = \cos\theta) \tag{1.1}$$

Here r is the dimensionless radial coordinate referred to the radius of the particle, θ the angular coordinate, Δ the axisymmetric spherical Laplace operator, D the diffusion coefficient, ψ the dimensionless stream function, and c the concentration of diffusing material. The boundary conditions have the form:

at infinity

$$r \to \infty, \quad \xi \to 0 \tag{1.2}$$

and on the surface of the sphere

$$r = 1, \quad \partial\xi / \partial r = k(\xi - 1), \quad k = ak_* D^{-1} \tag{1.3}$$

where k_* is the constant speed of chemical reaction on the surface of the particle.

In writing the condition (1.3) it is assumed that the absorption of material by the sphere depends on a surface chemical reaction of first order. For $k \to \infty$ the condition (1.3) takes the form $\xi(1) = 0$, and corresponds to the diffusion regime for absorption of material, the deposition of aerosol particles, and also heat transfer from the stream to a sphere whose temperature T_1 is maintained constant. (In the last case D stands for the coefficient of thermal conductivity, and ξ for the ratio $(T - T_0)\times(T_1 - T_0)^{-1}$, where T is the temperature of the stream and T_0 the temperature at infinity).

If the stream function $\psi = \psi(r, \mu)$ is known, Eq. (1.1) and the conditions (1.2) and (1.3) completely determine the distribution of concentration in the stream. Exact solution of the problem (1.1)–(1.3) is impossible, even if the velocity distribution for viscous flow past the sphere is taken as the simplest known approximate solution, the Stokes solution. Below an approximate analytical solution of (1.1)–(1.3) will be found by the method of matched asymptotic expansions with respect to the Péclet number P in the inner $(1 < r < P^{-1})$ and outer $(P^{-1} < r < \infty)$ flow regions. The basic features of the method, as applied to a variety of problems in gas dynamics and hydrodynamics, are described in [7]. For the velocity distribution in the inner and outer regions we use the solution obtained in [3] by the method of matched asymptotic expansions (see also [8]).

We seek inner and outer expansions respectively in the form

$$\xi_* = \sum_{n=0}^{\infty} \alpha_n(P)\,\xi_n(r, \mu) \tag{1.4}$$

$$\xi^* = \sum_{n=0}^{\infty} \alpha^{(n)}(P)\,\xi^{(n)}(\rho, \mu) \tag{1.5}$$

Here it is assumed with respect to the functions α_n (P) and $\alpha^{(n)}(P)$ simply that

$$\frac{\alpha_{n+1}(P)}{\alpha_n(P)} \to 0, \qquad \frac{\alpha^{(n+1)}(P)}{\alpha^{(n)}(P)} \to 0 \quad \text{as} \quad P \to 0$$

Terms of the expansion (1.4) are determined successively as solutions of Eq. (1.1) and boundary condition (1.3). Here the velocity field in (1.1) is given by the three-term inner expansion for the stream function [3]

$$\psi_* = \frac{1}{4}(r-1)^2(1-\mu^2)\left[\left(1+\frac{3}{8S}P+\frac{9}{40S^2}P^2\ln P\right)\left(2+\frac{1}{r}\right)-\right.$$
$$\left.-\frac{3}{8S}P\left(2+\frac{1}{r}+\frac{1}{r^2}\right)\mu\right]+O(P^2)\left(S=\frac{\nu}{D}\right) \qquad (1.6)$$

Here S is the Schmidt number, and ν the coefficient of kinematic viscosity. The outer expansion (1.5) is determined from Eq. (1.1) with the stream function ψ given by the two-term outer expansion [3], and the condition (1.2). Introducing the contracted radial coordinate $\rho = rP$ and $\psi^* = \psi P^2$, we rewrite (1.1) and (1.2) in the form

$$\Delta^*\xi^* = \frac{1}{\rho^2}\frac{\partial(\psi^*, \xi^*)}{\partial(\rho, \mu)}, \quad \rho\to\infty, \quad \xi^*\to 0$$
$$\psi^* = \frac{1}{2}\rho^2(1-\mu^2) - \frac{3}{2}SP(1+\mu)\left[1-\exp\left(-\frac{\rho}{S}\frac{1-\mu}{2}\right)\right]+O(P^2) \qquad (1.7)$$

Here Δ^* is the axisymmetric spherical Laplace operator obtained from Δ by replacing r with ρ. The arbitrary constants arising in the solution of (1.1), (1.3) and (1.6) and of (1.7) are determined by matching the inner and outer expansions (1.4) and (1.5).

2. Zeroth approximation. Construction of the solution begins with determination of the zeroth term of the outer expansion (1.5). In this case the problem (1.7) is obviously satisfied by the solution

$$\xi^{(0)} = 0 \qquad (2.1)$$

We now find the zeroth term of the inner expansion (1.4). From (1.1), (1.3) and (1.6) we have for $P = 0$

$$\Delta\xi_0 = 0, \quad r = 1, \quad \partial\xi_0/\partial r = k(\xi_0 - 1) \qquad (2.2)$$

The general solution of the problem (2.2) can be given in the form

$$\xi_0 = \frac{q}{r} + \sum_{n=0}^{\infty} a_n\left(r^n + \frac{n-k}{n+1+k}r^{-n-1}\right)P_n(\mu), \quad q = \frac{k}{k+1} \qquad (2.3)$$

The expression (2.3) contains arbitrary constants a_n that should be determined by matching (2.3) with (2.1). To match, the outer expansion should be expanded in powers of ρ. Then the constants are determined by requiring agreement in the behavior of terms in that series as $\rho \to 0$ and of terms in the expansion (2.3) as $r \to \infty$. The matching is trivial for the zeroth expansion; we obtain $a_n = 0$ $(n = 0, 1, \ldots)$. Consequently

$$\xi_0 = qr^{-1} \qquad (2.4)$$

3. First approximation. We first determine in explicit form the coefficient of $\alpha^{(1)}(P)$ in outer expansion. For this purpose we transform the solution (2.4) to outer variables. Then it follows from (2.4) that $\alpha^{(1)} = P$, so that the first approximation for the outer expansion is to be sought in the form

$$\xi^{*(1)} = P\xi^{(1)} \qquad (3.1)$$

Substituting (3.1) into (1.7) and retaining terms of order P we obtain

$$\Lambda\xi^{(1)} = 0, \qquad \Lambda = \Delta^* - \mu\frac{\partial}{\partial\rho} - \frac{1-\mu^2}{\rho}\frac{\partial}{\partial\mu}, \tag{3.2}$$
$$\rho \to \infty, \quad \xi^{(1)} \to 0$$

The general solution of the problem (3.2) has the form

$$\xi^{(1)} = \exp\frac{\rho\mu}{2}\left(\frac{\pi}{\rho}\right)^{1/2}\sum_{n=0}^{\infty} A_n K_{n+1/2}(1/2\rho) P_n(\mu)$$
$$K_{n+1/2}(1/2\rho) = \left(\frac{\pi}{\rho}\right)^{1/2}\exp\frac{-\rho}{2}\sum_{m=0}^{n}\frac{(n+m)!}{(n-m)!\,m!\rho^m} \tag{3.3}$$

Here the $K_{n+1/2}$ are MacDonald functions. The constants A_n are to be determined by matching, which in this case consists in comparing the behavior of the function (3.1) for $\rho \to 0$ and the function (2.4) for $r \to \infty$. It is easy to show that $A_0 = q\pi^{-1}$ and $A_n = 0$ $(n = 1, 2, \ldots)$. Consequently

$$\xi^{(1)} = q\rho^{-1}\exp\left[1/2\rho(\mu - 1)\right] \tag{3.4}$$

We find the first approximation for the inner expansion. For this we convert the expression for the function $\xi^{*(1)}$ to the inner variable r and represent it as a series in P. Then from (3.1) and (3.4) we find that $\alpha_1(P) = P$. Consequently it follows that the first approximation for the inner expansion is to be sought in the form

$$\xi_{*1} = \xi_0 + P\xi_1 \tag{3.5}$$

Substituting (3.5) into (1.1) and (1.3), using (1.6), and keeping terms of order P, we obtain the equation and boundary conditions for ξ_1

$$\Delta\xi_1 = -\frac{q}{r^2}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\mu \tag{3.6}$$

$$r = 1, \quad \partial\xi_1 / \partial r = k\xi_1 \tag{3.7}$$

The solution of Eq. (3.6) can be represented in the form

$$\xi_1 = q\left(\frac{1}{2} - \frac{3}{4r} - \frac{1}{8r^3}\right)\mu + \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-n-1}) P_n(\mu) \tag{3.8}$$

The boundary conditions (3.7) permit linear relationships to be established between the constants a_n and b_n $\quad (1-k)\,a_1 + 3/8q\,(k+3) = (2+k)\,b_1$

$$(n-k)\,a_n = (n+1+k)\,b_n \qquad (n = 0, 2, 3, \ldots) \tag{3.9}$$

To obtain explicit expressions for the coefficients in (3.8) we perform the matching of the expression (3.5) as $r \to \infty$ and (3.1) as $\rho \to 0$. Using (3.4), (3.8) and (3.9), we obtain

$$a_0 = -1/2q, \quad b_0 = 1/2q^2, \quad a_1 = 0, \quad b_1 = 3/8\,(k+3)(k+2)^{-1}q$$
$$a_n = b_n = 0 \qquad (n = 2, 3, \ldots)$$

Consequently

$$\xi_1 = -\frac{q}{2} + \frac{q^2}{2r} + q\left(\frac{1}{2} - \frac{3}{4r} + \frac{3}{8}\frac{k+3}{k+2}\frac{1}{r^2} - \frac{1}{8r^3}\right)\mu \tag{3.10}$$

4. Second approximation for outer expansion. The two-term inner expansion ξ_{*1} in outer variables has, on the basis of (3.5), (2.4) and (3.10), the form

$$\xi_{*1} = P\frac{q}{\rho} - Pq\frac{1-\mu}{2} + P^2\frac{q}{4\rho}(2q-3\mu)+\ldots \qquad (4.1)$$

From (4.1) it follows that $\alpha^{(2)}(P) = P^2$. Substituting the three-term outer expansion

$$\xi^{*(2)} = \xi^{(0)} + P\xi^{(1)} + P^2\xi^{(2)},$$

where $\xi^{(0)}$ and $\xi^{(1)}$ are given by Eqs. (2.1) and (3.4), into (1.7), we obtain for $\xi^{(2)}$

$$\Lambda\xi^{(2)} = \frac{3}{4}\frac{qS}{\rho^3}\left(\frac{S+1}{S} + \frac{2}{\rho} - \frac{S-1}{S}\mu\right)\exp\left(\rho\frac{S+1}{S}\frac{\mu-1}{2}\right) -$$

$$-\frac{3}{4}\frac{qS}{\rho^3}\left(1+\frac{2}{\rho}-\mu\right)\exp\left(\rho\frac{\mu-1}{2}\right) \qquad (4.2)$$

$$\rho\to\infty, \qquad \xi^{(2)}\to 0$$

To solve the problem (4.2) we make the substitution

$$\xi^{(2)}(\rho,\mu) = \exp\left({}^1/_2\rho\mu\right)\xi'^{(2)}(\rho,\mu)$$

As a result, Eq. (4.2) is transformed into the nonhomogeneous Helmholtz equation, The right-hand side of this equation is expanded in a series of Legendre polynomials $P_n(\mu)$, and its solution is also sought as a series in $P_n(\mu)$. After some calculation we obtain

$$\xi^{(2)}(\rho,\mu) = q\rho^{-1/2}\exp\left(\frac{\rho\mu}{2}\right)\sum_{n=0}^{\infty}\eta_n(\rho)P_n(\mu)$$

$$\eta_n(\rho) = K_{n+1/2}\left(\frac{\rho}{2}\right)\int_{\rho}^{C_n} I_{n+1/2}\left(\frac{\rho}{2}\right)L_n(\rho)\,d\rho - I_{n+1/2}\left(\frac{\rho}{2}\right)\int_{\rho}^{\infty} K_{n+1/2}\left(\frac{\rho}{2}\right)L_n(\rho)\,d\rho$$

$$L_n(\rho) = -\frac{3}{2}S\rho^{-5/2}\exp\left(-\frac{\rho}{2}\right)\left[\left(1+\frac{\rho}{2}\right)\delta_{n0} - \frac{\rho}{2}\delta_{n1}\right] + \qquad (4.3)$$

$$+\frac{3(2n+1)}{2}\pi^{1/2}S^{3/2}\rho^{-3}\exp\left(-\rho\frac{S+1}{2S}\right)\left[\left(1+\rho\frac{S+1}{2S} - nS + n\right)I_{n+1/2}\left(\frac{\rho}{2S}\right) -\right.$$

$$\left.-\rho\frac{S-1}{2S}I_{n+3/2}\left(\frac{\rho}{2S}\right)\right]$$

$$\delta_{ni} = \begin{cases}0, & n\neq i\\ 1, & n=i\end{cases}$$

Here $I_{n+1/2}$ is the modified Bessel function.

The constants C_n are determined by matching the solution $\xi^{*(2)}(\rho,\mu)$ with (4.1). For this it is necessary to expand the function $\xi^{(2)}(\rho,\mu)$ in series for small ρ. In writing the series we use the expansion of the function $\eta_n(\rho)$ in series with respect to ρ, which has the form

$$\eta_n(\rho) = [A_n K_{n+1/2}\left(\frac{\rho}{2}\right) + \rho^{1/2}\sum_{i=0}^{2}\delta_{ni}R_i(\rho) + O(\rho^{3/2})$$

$$R_0(\rho) = \frac{f(S)}{\rho} - \frac{\ln\rho}{2} + \frac{f(S)}{2} + \frac{\ln S}{2} + \frac{7}{6} - \frac{\gamma}{2} \qquad (4.4)$$

$$R_1(\rho) = -\frac{3}{4\rho} + \frac{3}{16S} - \left(\frac{1}{8} + \frac{1}{20S} - \frac{1}{40S^2}\right)\rho\ln\rho$$

$$R_2(\rho) = {}^1/_{24} - {}^1/_{16}S^{-1}$$

$$f(S) = -{}^1/_4 S^2 + {}^1/_8 S + {}^1/_4(S+1)^2(S-2)\ln(1+S^{-1}) - {}^1/_2\ln S$$

Here the A_n are new constants uniquely related to the C_n, and $\gamma = 0.577215\ldots$ is Euler's constant. The series (4.4) can be extended indefinitely, but the terms given above suffice for carrying out the matching. After matching $\xi^{*(2)}_s(\rho, \mu)$ with $\xi_{*1}(r, \mu)$ we find the constants A_n

$$A_0 = \pi^{-1/2}[{}^1/_2 q - f(s)], \qquad A_n = 0 \quad (n \geqslant 1) \tag{4.5}$$

With these values of A_n the outer asymptotic expansion matches with the inner with accuracy up to terms of order P^2. The explicit expression for the asymptotics of the function $\xi^{(2)}(\rho, \mu)$ has the form

$$\xi^{(2)}(\rho, \mu) = \frac{q}{2\rho}\left(q - \frac{3}{2}\mu\right) - \frac{q}{2}\ln q + q\zeta(q, S) + \tag{4.6}$$

$$+ \frac{q}{4}\left[1 + \frac{3}{4S} - \left(\frac{3}{2} + \frac{1}{5S} - \frac{1}{10S^2}\right)\rho\ln\rho\right]\mu - \frac{5q}{24}\left(1 + \frac{3}{10S}\right)\frac{3\mu^2 - 1}{2} + O(\rho)$$

$$\zeta(q, S) = -{}^1/_4 S^2 + {}^1/_8 S + {}^{25}/_{24} - {}^1/_4 q - {}^1/_2\gamma + {}^1/_4 (S + 1)^2 (S - 2)\ln(1 + S^{-1})$$

5. Second and third approximations for inner expansion. It is evident from Eq. (4.6) that logarithmic as well as algebraic singularities appear in the second approximation for the outer expansion. The splitting of the singularities occurs also in the hydrodynamic problem of flow past the body. Each singularity generates a corresponding term in the inner expansion. It turns out to be possible to determine at once the next two approximations for the inner expansion. It will be evident below that the splitting of singularities is perpetuated in subsequent steps of the solution. Transforming (4.6) to inner variables, we determine the coefficient $\alpha_2(P)$ of the inner expansion (1.4), obtaining

$$\alpha_2(P) = P^2 \ln P \tag{5.1}$$

From (1.1) and (1.3) we find that the function $\xi_2(r, \mu)$ satisfies the Laplace equation with homogeneous boundary conditions

$$\Delta\xi_2 = 0, \quad r = 1, \quad \partial\xi_2 / \partial r = k\xi_2 \tag{5.2}$$

The general solution of the problem (5.2) has the form

$$\xi_2 = \sum_{n=0}^{\infty} a_n \left(r^n + \frac{n-k}{n+1+k}\frac{1}{r^{n+1}}\right) P_n(\mu)$$

The constants a_n are determined by matching $\xi_{*2}(r, \mu)$ as $r \to \infty$ with $\xi^{*(2)}(\rho, \mu)$ as $\rho \to 0$. We find

$$a_0 = -{}^1/_2 q, \qquad a_r = 0 \qquad (n \geqslant 1)$$

Thus

$$\xi_2 = {}^1/_2 q\,(qr^{-1} - 1) \tag{5.3}$$

As follows from (4.6), the third approximation for the inner expansion should be of order

$$\alpha_3(P) = P^2 \tag{5.4}$$

The function $\xi_3(r, \mu)$ satisfies the equation

$$\Delta\xi_3 = q\sum_{n=0}^{2} Z_n(r) P_n(\mu)$$

$$Z_0(r) = \frac{1}{3r} - \frac{1}{2r^2} + \frac{1}{48}\frac{23 + 7k}{2 + k}\frac{1}{r^4} + \frac{1}{8r^5} - \frac{3}{16}\frac{3 + k}{2 + k}\frac{1}{r^6} + \frac{1}{12r^7}$$

$$Z_1(r) = \left(\frac{q}{4} + \frac{3}{16S}\right)\left(-\frac{2}{r^2} + \frac{3}{r^3} - \frac{1}{r^5}\right)$$

$$Z_2(r) = -\frac{1}{3r} + \frac{5}{4r^2} - \frac{3}{8}\frac{12+5k}{2+k}\frac{1}{r^3} + \frac{5}{48}\frac{35+13k}{2+k}\frac{1}{r^4} - \frac{5}{16r^5} - \frac{3}{16} \times$$
$$\times \frac{3+k}{2+k}\frac{1}{r^6} + \frac{5}{48r^7} + \frac{3}{16S}\left(\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^4} - \frac{1}{r^5} + \frac{1}{r^6}\right) \quad (5.5)$$

The boundary condition is

$$r = 1, \quad \partial\xi_3 / \partial r = k\xi_3 \quad (5.6)$$

The general solution of Eq. (5.5) has the form

$$\xi_3 = q \sum_{n=0}^{\infty} [\xi_{3,n}(r) + a_n r^n + b_n r^{-n-1}] P_n(\mu)$$

$$\xi_{3,0}(r) = \frac{r}{6} - \frac{\ln r}{2} + \frac{7k+23}{k+2}\frac{1}{96r^2} + \frac{1}{48r^3} - \frac{k+3}{k+2}\frac{1}{64r^4} + \frac{1}{240r^5}$$

$$\xi_{3,1}(r) = \left(q + \frac{3}{4S}\right)\left(\frac{1}{4} - \frac{3}{8r} - \frac{1}{16r^3}\right)$$

$$\xi_{3,2}(r) = \frac{r}{12} - \frac{5}{24} + \frac{5k+12}{k+2}\frac{1}{16r} - \frac{13k+35}{k+2}\frac{5}{192r^2} + \frac{\ln r}{16r^3} - \frac{k+3}{k+2} \times$$
$$\times \frac{1}{32r^4} + \frac{5}{672r^5} + \frac{3}{16S}\left(-\frac{1}{3} + \frac{1}{2r} - \frac{1}{4r^2} + \frac{\ln r}{5r^3} + \frac{1}{6r^4}\right)$$
$$\xi_{3,n}(r) \equiv 0, \quad n \geqslant 3 \quad (5.7)$$

By virtue of the boundary condition (5.6) the constants a_n and b_n are connected by the linear relationship

$$(n-k)a_n + \xi_{3,n}'(1) - k\xi_{3,n}(1) = (n+1+k)b_n \quad (5.8)$$

To determine a_n amd b_n it is necessary to carry out the matching of the inner and outer asymptotic expansions. For this we transform (5.7) to outer variables. Taking account of (1.4), (4.1), (5.1), (5.3), (5.4), (5.7) and (5.8), we obtain

$$\xi_{*3} = q \sum_{n=2}^{\infty} a_n \rho^n P_n(\mu) P^{-n+2} + q\left[\rho^{-1} - {}^1/_2 + \frac{\rho}{6} + ({}^1/_2 + a_1\rho) P_1(\mu) + {}^1/_{12}\rho P_2 \times\right.$$
$$\left.\times (\mu)\right] P + q\left[\frac{q}{2\rho} - \frac{\ln\rho}{2} + a_0 + \frac{1}{4}\left(q + \frac{3}{4S}\right)P_1(\mu) - \left(\frac{5}{24} + \frac{1}{16S}\right)P_2(\mu)\right] \times$$
$$\times P^2 + \frac{q^2}{2\rho} P^3 \ln P + O(P^3) \quad (5.9)$$

It is evident from (5.9) that the inner expansion ξ_{*3} matches to terms of order P^2 with the outer expansion $\xi^{*(2)}$ determined by Eqs. (1.5), (3.1), (3.4) and (4.3) if the constants a_n are

$$a_0 = \zeta(q, S), \quad a_1 = -{}^1/_4, \quad a_n = 0 \quad (n \geqslant 2) \quad (5.10)$$

We find the constants b_n by using the relations (5.7), (5.8) and (5.10), and obtain

$$b_0 = -q\zeta(q, S) - \frac{239}{960} - \frac{79}{240}\frac{1}{k+1} + \frac{1}{32}\frac{1}{(k+1)(k+2)}$$

$$b_1 = \frac{7}{16} + \frac{9}{64S}\left(1 + \frac{1}{k+2}\right) - \frac{3}{4}\frac{1}{k+2} - \frac{3}{8}\frac{1}{(k+1)(k+2)}$$

$$b_2 = \frac{235}{1344} - \frac{1}{64S}\left(1 + \frac{13}{5}\frac{1}{k+3}\right) + \frac{3}{14}\frac{1}{k+3} + \frac{3}{16}\frac{1}{(k+2)(k+3)}$$

$$b_n = 0 \quad (n \geqslant 3) \quad (5.11)$$

The function $\xi_3(r, \mu)$ is completely determined by Eqs. (5.7), (5.10) and (5.11).

6. Approximations of higher order. It follows from (5.9) that

$$\alpha^{(3)} = P^3 \ln P \tag{6.1}$$

Substituting the series (1.5) into the relation (1.7), and considering Eqs. (2.1) and (6.1), leads to the conclusion that the unknown function $\xi^{(3)}(\rho, \mu)$ satisfies the same equation and boundary condition as the function $\xi^{(1)}(\rho, \mu)$. These functions consequently agree to within a multiplier, whose value is found by matching $\xi^{*(3)}(\rho, \mu)$ with (5.9). Using Eq. (3.4), we find after matching

$$\xi^{(3)} = \frac{q^2}{2\rho} e^{1/2\rho(\mu-1)} \tag{6.2}$$

From (6.1) and (6.2) it follows that

$$\alpha_4 = P^3 \ln P \tag{6.3}$$

After substituting the series (1.4) into (1.1) and (1.3), with regard to (1.6), we obtain the equation and boundary condition for ξ_4

$$\Delta\xi_4 = -\left(\frac{q}{2} + \frac{9}{40S^2}\right)\frac{q}{r^2}\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\mu \tag{6.4}$$

$$r = 1, \quad \partial\xi_4 / \partial r = k\xi_4 \tag{6.5}$$

Equation (6.4) differs from (3.6) merely in the multiplier on the right-hand side, and the boundary conditions (6.5) and (3.7) agree. Hence using (3.8) and (3.9), we find for ξ_4

$$\xi_4 = a_0 - \frac{qa_0}{r} + \left[a_1 r + \frac{q}{2}\left(\frac{q}{2} + \frac{9}{40S^2}\right)\left(1 - \frac{3}{2r} + \frac{3}{4}\frac{k+3}{k+2}\frac{1}{r^2} - \frac{1}{4r^3}\right) - \right.$$
$$\left. - \frac{k}{k+2}\frac{a_1}{r^2}\right]\mu + \sum_{n=2}^{\infty} a_n \left(r^n + \frac{n-k}{n+1+k} r^{-n-1}\right) P_n(\mu) \tag{6.6}$$

The constants a_n are determined by matching the expansions for ξ_{*4} and $\xi^{*(3)}$. Transforming (6.6) to outer variables, we obtain after matching

$$a_0 = -{}^1/_4 q^2, \quad a_n = 0 \quad (n \geqslant 1) \tag{6.7}$$

It is evident from what is set forth above that the determination of the third approximation for the function $\xi^*(\rho, \mu)$ and the fourth for $\xi_*(r, \mu)$ does not involve tedious calculation thanks to the appearance of a logarithmic singularity. Obviously the search for approximations of still higher order will be associated with voluminous computations. In addition, the problem is complicated by the necessity of first determining higher terms in the inner and outer asymptotic expansions for the stream function, which represents an independent problem. We limit ourselves here merely to pointing out that, as follows from (1.6), (1.7), (5.9), (6.2), (6.3) and (6.6), the next approximations for $\xi^*(\rho, \mu)$ and $\xi_*(r, \mu)$ must have the order

$$\alpha^{(4)}(P) = \alpha_5(P) = P^3 \tag{6.8}$$

Furthermore, as is evident from (6.2) and (6.6), the function $\xi_5(r, \mu)$ will contain terms of the form ${}^9/_{80}\, \mu q S^{-2} \ln r$, which leads to complete matching of $\xi_{*5}(r, \mu)$ with $\xi^{*(4)}(\rho, \mu)$.

7. Concentration (temperature) field. Flux of material (heat) at surface of particle. Summarizing the results obtained above, we write the expression for the distribution of concentration (or temperature) in the stream flowing past a sphere.

Far from the sphere (the outer asymptotic expansion) we have

$$\xi^* = \frac{q}{\rho} e^{1/2\rho(\mu-1)}\left(P + \frac{q}{2}P^3\ln P\right) + \xi^{(2)}P^2 + O(P^3) \tag{7.1}$$

Here the function $\xi^{(2)} = \xi^{(2)}(\rho, \mu)$ is given by Eq. (4.6). Near the sphere (the inner asymptotic expansion) we have

$$\xi_* = \frac{q}{r} + \frac{q}{2}\left(\frac{q}{r} - 1\right)\left(P + P^2\ln P + \frac{q}{2}P^3\ln P\right) + \frac{q}{2}\left(1 - \frac{3}{2r} + \frac{3}{4}\times \frac{k+3}{k+2}\frac{1}{r^2} - \frac{1}{4r^3}\right)\mu\left[P + \left(\frac{q}{2} + \frac{9}{40S^2}\right)P^3\ln P\right] + q\left[\xi_{3,0} + \zeta + \frac{b_0}{r} + \left(\xi_{3,1} - \frac{r}{4} + \frac{b_1}{r^2}\right)\mu + \left(\xi_{3,2} + \frac{b_2}{r^2}\right)\frac{3\mu^2 - 1}{2}\right]P^2 + O(P^3) \tag{7.2}$$

Here $\xi_{3,n}$ and b_n are given by Eqs. (5.11). We note that in the special case $k \to \infty$, $S \to \infty$ $(R \to 0)$ the expressions (7.1) and (7.2) agree with those obtained in [5].

The heat (or mass) exchange with the medium is usually characterized by the Nusselt number N. Taking the diameter of the particle as characteristic length, we have for the Nusselt number

$$N = -\int_{-1}^{1}\left(\frac{\partial \xi_*}{\partial r}\right)_{r=1} d\mu \tag{7.3}$$

After integration we find

$$N = 2q + q^2\left(P + P^2\ln P + {}^1/_2 qP^3\ln P\right) + qQ(q, S)P^2 + O(P^3) \tag{7.4}$$

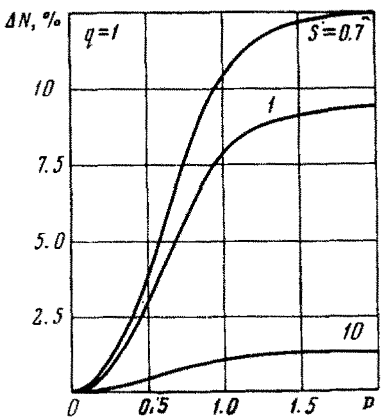

Fig. 2

$$Q(q, S) = {}^1/_2 q^2 - {}^{119}/_{80} q + {}^3/_{32} + \gamma - {}^3/_{16}(2 - q)^{-1} + {}^1/_2 S^2 - {}^1/_4 S - {}^1/_2 (S + 1)^2 (S - 2)\ln(1 + S^{-1})$$

The results obtained indicate a significant dependence of the total flux of matter (or heat) at the surface of the particle upon the Reynolds number. Fig. 2 shows the relative increment in Nusselt number $\Delta N = N(S, P)\, N^{-1}(\infty, P) - 1$ for finite R with $q = 1$ and various values of S and P. It is evident, for example, that for $S \leqslant 0.7$ and $P \geqslant 1$ this increment exceeds 10%. The effect of reaction speed on Nusselt number is shown in Fig. 3 for $S = 1$ and various Reynolds numbers.

In conclusion we note that the results obtained above are valid for any reaction speed at the surface of the particle (any value of q in the range $0 \leqslant q \leqslant 1$) and any value of the Schmidt number S that is not small (the case $S \ll 1$ being of little practical interest). It is evident that the method of matched asymptotic expansions does not permit establishing admissible upper limits on the Péclet number. This question can be solved only by comparison with experiment.

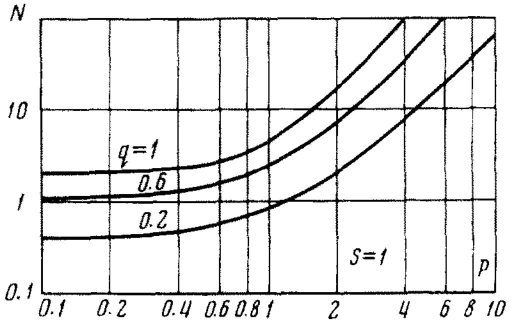

Fig. 3

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